

Physics, stability, and dynamics of supply networks

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We show how to treat supply networks as physical transport problems governed by balance equations and equations for the adaptation of production speeds. Although the nonlinear behavior is different, the linearized set of coupled differential equations is formally related to those of mechanical or electrical oscillator networks. Supply networks possess interesting features due to their complex topology and directed links. We derive analytical conditions for absolute and convective instabilities. The empirically observed “bullwhip effect” in supply chains is explained as a form of convective instability based on resonance effects. Moreover, it is generalized to arbitrary supply networks. Their related eigenvalues are usually complex, depending on the network structure (even without loops). Therefore, their generic behavior is characterized by damped or growing oscillations. We also show that regular distribution networks possess two negative eigenvalues only, but perturbations generate a spectrum of complex eigenvalues.

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Econophysics has stimulated a lot of interesting research on problems in finance and economic systems [1], applying methods from statistical physics and the theory of complex systems. Recently, the dynamics of supply networks [2–4], i.e., the flow of materials through networks, has been identified as an interesting physical transport problem [3–5], which is also reflected by the notion of “factory physics” [6]. Potential applications reach from production networks to business cycles, from metabolic networks to food webs, up to logistic problems in disaster management [7]. Empirical and theoretical studies have shown that the flow of goods between different producers or suppliers can be described analogously to driven many-particle flows between sources and sinks (depots), where the particles represent materials, goods, or other resources. In contrast to the stationary behavior assumed by most production engineers, this flow may display a complex dynamics in time, including oscillatory patterns and chaos [8,9]. A particular focus has been on the empirically observed and well-known *bullwhip effect* [3,5,9–11], which describes the amplification of the oscillation amplitudes of delivery rates in supply chains compared to the variations in the consumption rate of goods.

A promising approach to the nonlinear interactions and dynamics of supply chains is based on fluid-dynamic models [5,10], which are related to macroscopic traffic models [3–5]. In contrast to classical approaches like queuing theory and event-driven simulations, they are better suited for on-line control under dynamically changing conditions. These fluid-dynamic models have recently been generalized to cope with

discrete spaces (production steps) [3,5]. However, the impact of the topology of supply networks, connecting the subject to the statistical physics of networks [12], has not yet been thoroughly investigated. Its relevance for the stability and dynamics of supply networks will, therefore, be the main focus of this paper. Here, we will specifically concentrate on analytical results for the dynamic behavior in the linear regime around the usually assumed stationary state, while nonlinear effects are investigated elsewhere [3]. The corresponding equations can be mapped onto the ones of particular mechanical or electrical networks, but supply networks are generally more complex and their nonlinear dynamics is different. In particular, supply networks are directed and possess other characteristic topologies [13]. Hence, our analytical investigation yields interesting results. Apart from a generalization of the bullwhip effect from sequential supply chains to networks, they allow us to explain various surprising features of numerical simulations of supply networks.

(i) The bullwhip effect can occur even though the eigenvalues of sequential supply chains are negative.

(ii) Supply *networks* tend to show oscillations, even without loops in the material flows.

(iii) Regular distribution networks are characterized by two negative eigenvalues, but random perturbations in the network structure cause a spectrum of oscillating modes.

Our simplified model of supply networks consists of u suppliers i delivering products to other suppliers j or consumers. The suppliers j (e.g., production units of a plant or company) will be assumed to deliver products, resources, or materials k at a certain rate $d_{kj}X_j(t)$. Their demand of goods of kind i from other suppliers j is given by $c_{ij}X_j(t)$, where $0 \leq c_{ij} \leq 1$. The proportionality $c_{ij}X_j(t)$ to the production rate $X_j(t)$ of products j reflects that a quantity c_{ij} of product i is

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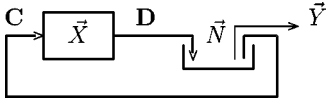


FIG. 1. Schematic illustration of the structure of the proposed supply network model: Suppliers (box) deliver commodities according to the output matrix \mathbf{D} and thereby increase the market's inventory level \vec{N} (bowl). Commodities available on the market can either reenter other suppliers according to the input matrix \mathbf{C} or leave the cycle for final consumption \vec{Y} .

needed to produce one unit of product j . The coefficients c_{ij} define an input matrix $\mathbf{C}=(c_{ij})$ and d_{ij} an output matrix \mathbf{D} . The interactions among the different suppliers i and j are represented by the supply matrix

$$\mathbf{S} = \mathbf{D} - \mathbf{C}. \quad (1)$$

In addition, we need to consider the inventories, stock levels, or availability $N_i(t)$ of products, materials, or resources i .

Our model for the dynamics of supply networks consists of two sets of equations, one for the change of inventories $N_i(t)$ with time t and another one for the adaptation of the delivery or production rates $X_j(t)$ of the suppliers or production units j . The inventories $N_i(t)$ of goods of kind i are described by material balance equations for the flows of goods, which could be viewed as a discontinuous version of the continuity equation (see Fig. 1):

$$\frac{dN_i}{dt} = \underbrace{\sum_{k=1}^u d_{ik} X_k(t)}_{\text{supply}} - \underbrace{\left[\sum_{j=1}^u c_{ij} X_j(t) + Y_i(t) \right]}_{\text{demand}}. \quad (2)$$

Additionally, we have

$$0 \leq c_{ij}, d_{ij} \leq 1 \quad (1 \leq i, j \leq u) \quad (3)$$

and the “normalization conditions”

$$d_{i0} = 1 - \sum_{k=1}^u d_{ik} \geq 0, \quad (4)$$

$$c_{i,u+1} = 1 - \sum_{j=1}^u c_{ij} \geq 0.$$

Frequently, one assumes $d_{ij} = \delta_{ij}$ with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise [3]. This corresponds to situations in which production units or suppliers are defined such that they deliver one kind of good only. The quantity

$$Y_i(t) = \underbrace{c_{i,u+1} X_{u+1}(t)}_{\text{consumption and losses}} - \underbrace{d_{i0} X_0(t)}_{\text{inflow of resources}} \quad (5)$$

comprises the consumption rate of goods i , losses, and waste (the “export” of material), minus the inflows into the considered system (the “imports”).

Changes in the demand $Y_i(t)$ sooner or later require an adaptation of the production rates $X_j(t)$. This adaptation takes

time (to determine the change in demand, to finish the planning process and administrative steps, and to adapt orders and production capacities). We will reflect this by an adaptation time T and assume that the change in the production rate is proportional to the deviation of the actual production rate X_j from the desired one $W_j(\{N_{ij}, \{dN_i/dt\})$, which depends on the inventories N_i and their changes dN_i/dt in time:

$$\frac{dX_j}{dt} = \frac{1}{T} \left[W_j \left(\{N_{ij}, \left\{ \frac{dN_i}{dt} \right\} \right) - X_j(t) \right]. \quad (6)$$

Although a more general treatment is possible, we will first focus on cases where the delivery rate X_j of supplier j is adapted only to the inventory N_i of the dominant product i given by $d_{ij} = \max_k d_{kj}$. Moreover, we will enumerate suppliers j according to their dominating products i , which implies $j = i$. The resulting adaptation equation

$$\frac{dX_i}{dt} = \frac{1}{T} \left[W_i \left(N_i, \frac{dN_i}{dt} \right) - X_i(t) \right] \quad (7)$$

still implies an indirect dependence on the production and delivery rates X_j of other suppliers $j \neq i$ via the dependence on dN_i/dt [see Eq. (2)].

In order to gain analytical results, we will assume for the desired production rate $W_i(N_i, dN_i/dt)$ that (i) it is non-negative and decreasing with increasing inventories, (ii) it is proportional to the steady-state inventory \bar{N}_i , and (iii) it responds to the relative deviation N_i/\bar{N}_i of the inventory N_i from the steady-state inventory \bar{N}_i and to the relative change $(dN_i/dt)/N_i(t)$ of the inventory in time. This suggests the scaling relation

$$W_i \left(N_i, \frac{dN_i}{dt} \right) \approx \nu \bar{N}_i P \left(\frac{N_i}{\bar{N}_i}, \frac{1}{N_i} \frac{dN_i}{dt} \right) \quad (8)$$

with a scaling function P which, for simplicity, is assumed to be identical for all sectors. The stationary value \bar{N}_i of the inventory N_i can be easily determined by solving the linear system of equations

$$\bar{Y}_i = \sum_{j=1}^u (d_{ij} - c_{ij}) \bar{X}_j = \nu P(1, 0) \sum_{j=1}^u (d_{ij} - c_{ij}) \bar{N}_j, \quad (9)$$

where \bar{Y}_i denotes the time average of the demand $Y_i(t)$, while \bar{X}_i and \bar{N}_i denote the stationary values corresponding to the case $Y_i = \bar{Y}_i$. The stationary solution is, therefore, given by

$$\bar{N}_i = \frac{1}{\nu P(1, 0)} \sum_j (\mathbf{D} - \mathbf{C})_{ij}^{-1} \bar{Y}_j, \quad (10)$$

$$\bar{X}_i = W_i(\bar{N}_i, 0) = \nu \bar{N}_i P(1, 0),$$

where $(\mathbf{D} - \mathbf{C})^{-1}$ represents the inverse matrix of $(\mathbf{D} - \mathbf{C})$. Note that, by appropriate choice of ν , one can set $P(1, 0) = 1$.

Since we want to know whether the stationary state of the supply network is stable, as is often implicitly assumed in queuing theory, we will focus on what happens in the case of small deviations

$$\begin{aligned}
x_i(\tau) &= [X_i(\tau T) - \bar{X}_i]T, \\
n_i(\tau) &= N_i(\tau T) - \bar{N}_i, \\
y_i(\tau) &= [Y_i(\tau T) - \bar{Y}_i]T
\end{aligned} \tag{11}$$

from it. Here, we have introduced a scaling of the model variables to dimensionless units. For example, $\tau = t/T$ is a dimensionless time, i.e., in the following we will measure the time in units of T . Close to equilibrium, we can linearize the model equations. In matrix notation they read

$$\frac{d}{d\tau} \begin{pmatrix} \vec{n}(\tau) \\ \vec{x}(\tau) \end{pmatrix} = \mathbf{M} \begin{pmatrix} \vec{n}(\tau) \\ \vec{x}(\tau) \end{pmatrix} + \begin{pmatrix} -\vec{y}(\tau) \\ B\vec{y}(\tau) \end{pmatrix} \tag{12}$$

with

$$\mathbf{M} = \begin{pmatrix} \mathbf{0}, & \mathbf{S} \\ -A\mathbf{E}, & -\mathbf{E} - B\mathbf{S} \end{pmatrix}. \tag{13}$$

Here,

$$\begin{aligned}
A &= -\nu T \frac{\partial P(1,0)}{\partial z_1}, \\
BT &= -\nu T \frac{\partial P(1,0)}{\partial z_2}
\end{aligned} \tag{14}$$

denote the negative values of the partial derivatives of $TW_i(N_i, dN_i/dt)$ with respect to the first variable

$$z_1 = \frac{N_i}{N_i} \tag{15}$$

and the second one

$$z_2 = \frac{dN_i/dt}{N_i} = \frac{dN_i/d\tau}{N_i T} \tag{16}$$

in the equilibrium point $(N_i, dN_i/dt) = (\bar{N}_i, 0)$. The negative signs reflect that A and B are typically positive, as the production rate should decrease when the inventory increases.

The above linear system of coupled differential equations is a quite general approach to the study of supply networks and material flows close to the stationary state [14]. It describes the response of delivery rates to a variation $\vec{y}(\tau)$ in the consumption rate and can be derived from various non-linear supply chain models which differ in their degree of detail concerning the consideration of forecasts [4], price dynamics [3,13], or lack of materials [3]. Therefore, it is worth investigating the dependence on the dimensionless parameters A and B , and on the structure of the supply network characterized by \mathbf{S} . The system of $2u$ differential equations can also be written as a system of u second-order differential equations

$$\frac{d^2 \vec{x}}{d\tau^2} + [\mathbf{E} + B\mathbf{S}] \frac{d\vec{x}}{d\tau} + A\mathbf{S}\vec{x}(\tau) = \vec{g}(\tau), \tag{17}$$

where

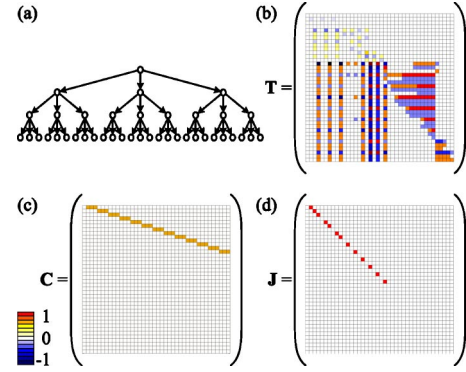


FIG. 2. (Color online) (a) Example of a regular distribution network. (b)–(d) Matrix \mathbf{T} allows one to transform the input matrix $\mathbf{C} = \mathbf{E} - \mathbf{S}$ into a matrix \mathbf{J} of Jordan normal form.

$$\vec{g}(\tau) = A\vec{y}(\tau) + B \frac{d\vec{y}}{d\tau}. \tag{18}$$

Note that there exists a matrix \mathbf{T} which allows one to transform the input matrix \mathbf{C} or, more generally, the matrix $\mathbf{E} - \mathbf{S} = \mathbf{C} + (\mathbf{E} - \mathbf{D})$ via

$$\mathbf{T}^{-1}(\mathbf{E} - \mathbf{S})\mathbf{T} = \mathbf{J} \tag{19}$$

into either a diagonal or a Jordan normal form \mathbf{J} (see Fig. 2).

Defining

$$\begin{aligned}
\vec{\mu}(\tau) &= \mathbf{T}^{-1}\vec{x}(\tau), \\
\vec{h}(\tau) &= \mathbf{T}^{-1}\vec{g}(\tau),
\end{aligned} \tag{20}$$

we obtain the coupled set of second-order differential equations

$$\frac{d^2 \mu_i}{d\tau^2} + 2\gamma_i \frac{d\mu_i}{d\tau} + \omega_i^2 \mu_i = b_i \left(B\mu_{i+1} + A \frac{d\mu_{i+1}}{d\tau} \right) + h_i, \tag{21}$$

where

$$\begin{aligned}
\gamma_i &= [1 + B(1 - J_{ii})]/2, \\
\omega_i &= [A(1 - J_{ii})]^{1/2}, \\
b_i &= J_{i,i+1}.
\end{aligned} \tag{22}$$

This can be interpreted as a set of equations for linearly coupled damped oscillators with damping constants γ_i , eigenfrequencies ω_i , and external forcing $h_i(\tau)$. The other forcing terms on the right-hand side are due to interactions of suppliers. They appear only if \mathbf{J} is not of diagonal but of Jordan normal form. Because of $b_i = J_{i,i+1}$, Eqs. (21) can always be analytically solved in a recursive way, starting with the highest index $i = u$. Note that, in the case $\mathbf{D} = \mathbf{E}$ (i.e., $d_{ij} = \delta_{ij}$), J_{ii} are the eigenvalues of the input matrix \mathbf{C} and

$$0 \leq |J_{ii}| \leq 1. \tag{23}$$

Equation (21) has a special periodic solution of the form

$$\mu_i(\tau) = \mu_i^0 e^{i(\alpha\tau - \beta t)},$$

$$h_i(\tau) = h_i^0 e^{i\alpha\tau}, \quad (24)$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Inserting this into Eq. (21) and dividing by $e^{i\alpha\tau}$ immediately gives

$$(-\alpha^2 + 2i\alpha\gamma_i + \omega_i^2)\mu_i^0 e^{-i\beta_i} = b_i(A + i\alpha B)\mu_{i+1}^0 e^{-i\beta_{i+1}} + h_i^0. \quad (25)$$

With $e^{\pm i\phi} = \cos(\phi) \pm i \sin(\phi)$ this implies

$$\begin{aligned} \mu_i^0 e^{-i\beta_i} &= \frac{b_i(A + i\alpha B)\mu_{i+1}^0 e^{-i\beta_{i+1}} + h_i^0}{-\alpha^2 + 2i\alpha\gamma_i + \omega_i^2} \\ &= \frac{\sqrt{\text{Re}^2 + \text{Im}^2} e^{i\phi_i}}{\sqrt{(\omega_i^2 - \alpha^2)^2 + (2\alpha\gamma_i)^2} e^{i\phi_i}}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \text{Re} &= b_i \mu_{i+1}^0 [A \cos(\beta_{i+1}) + \alpha B \sin(\beta_{i+1})] + h_i^0, \\ \text{Im} &= b_i \mu_{i+1}^0 [\alpha B \cos(\beta_{i+1}) - A \sin(\beta_{i+1})]. \end{aligned} \quad (27)$$

Moreover,

$$\mu_i^0 = \sqrt{\frac{[A^2 + (\alpha B)^2](b_i \mu_{i+1}^0)^2 + h_i^0 H_i + (h_i^0)^2}{(\omega_i^2 - \alpha^2)^2 + (2\alpha\gamma_i)^2}} \quad (28)$$

with

$$H_i = 2b_i \mu_{i+1}^0 [A \cos(\beta_{i+1}) + \alpha B \sin(\beta_{i+1})]. \quad (29)$$

Finally, we have

$$\beta_i = \varphi_i - \rho_i \quad (30)$$

with

$$\tan \varphi_i = \frac{2\alpha\gamma_i}{\omega_i^2 - \alpha^2} \quad (31)$$

and

$$\tan \rho_i = \frac{b_i \mu_{i+1}^0 [\alpha B \cos(\beta_{i+1}) - A \sin(\beta_{i+1})]}{b_i \mu_{i+1}^0 [A \cos(\beta_{i+1}) + \alpha B \sin(\beta_{i+1})] + h_i^0}. \quad (32)$$

For $h_i^0 = 0$, we obtain

$$\tan(\varphi_i - \beta_i) = \tan(\delta - \beta_{i+1}) \quad (33)$$

with

$$\tan \delta = \frac{\alpha B}{A}, \quad (34)$$

i.e., the phase shift between i and $i+1$ is just

$$\beta_i - \beta_{i+1} = \varphi_i - \delta. \quad (35)$$

According to Eq. (21), the dynamics of our supply network model can surprisingly be reduced to the dynamics of a linear (sequential) supply chain (see Fig. 3), but with the new entities i having the meaning of “quasisuppliers” (analogously to “quasispecies” [15,16]) defined by the linear combination $\vec{\mu}(\tau) = \mathbf{T}^{-1} \vec{x}(\tau)$. This transformation makes it possible to define the bullwhip effect for arbitrary supply networks: It occurs if the amplitude μ_i^0 is greater than the

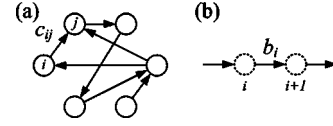


FIG. 3. In the proposed supply network model, an arbitrary network topology (a) can be transformed into a sequential supply chain (b) of “quasisuppliers” with potentially complex eigenvalues. The coupling b_i corresponds to the superdiagonal elements $J_{i,i+1}$, which are either 0 or 1. It is therefore possible to find several uncoupled chains of smaller size.

oscillation amplitude μ_{i+1}^0 of the next downstream supplier $i+1$ and greater than the amplitude h_i^0 of the external forcing, i.e., if

$$\frac{\mu_i^0}{\max(\mu_{i+1}^0, h_i^0)} > 1. \quad (36)$$

This *resonance effect* corresponds to the case of *convective instability*. According to formula (28), the bullwhip effect is particularly likely to appear, if the oscillation frequency α of the consumption rate is close to one of the resonance frequencies ω_i .

Depending on the respective network structure, it can also happen that the oscillation amplitude of $\vec{\mu}(\tau)$ is amplified in the course of time (see Fig. 4). This case of *absolute instability* can occur if at least one of the eigenvalues $\lambda_{i,\pm}$ of the homogeneous equation (21) resulting for $h_i = b_i = 0$ has a positive real part. The (up to) $2u$ eigenvalues

$$\lambda_{i,\pm} = -\gamma_i \pm \sqrt{\gamma_i^2 - \omega_i^2} \quad (37)$$

depend on the (quasi) supplier i and determine the temporal evolution of the amplitude of deviations from the stationary solution.

One can distinguish various interesting cases.

(i) If the supply matrix \mathbf{S} is symmetric, as for most mechanical or electrical oscillator networks, all eigenvalues J_{ii} are real. Consequently, if $\omega_i < \gamma_i$ (i.e., if A is small enough), the eigenvalues $\lambda_{i,\pm}$ of \mathbf{M} are real and negative, corresponding to an overdamped behavior. However, if B is too small, the system behavior may be characterized by damped oscillations.

(ii) Most natural and man-made supply networks have directed links, and \mathbf{S} is not symmetric. Therefore, some of the eigenvalues J_{ii} will normally be complex, and an overdamped behavior is untypical. The characteristic behavior is rather of oscillatory nature (although asymmetry does not always imply complex eigenvalues [13]). For small values of B , it can even happen that the real part of an eigenvalue $\lambda_{i,\pm}$

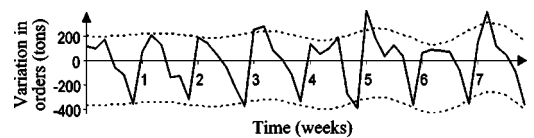


FIG. 4. Empirical example for a temporal amplification in the oscillation amplitude of the weekly order flow of a major company around its average ($\gamma = 0.05$, $\omega = 2\pi/T$ with $T = 2$ weeks).

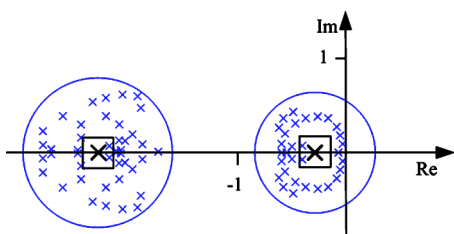


FIG. 5. Spectrum of a supply network [the regular distribution network displayed in Fig. 2(a)] with two degenerate real eigenvalues (squares). The eigenvalues $\lambda_{i,\pm}$ of the randomly perturbed supply network (crosses) are mostly complex and located within Geršgorin's disks (large circles).

becomes positive. This implies an amplification of the oscillations in time (until the oscillation amplitude is limited by nonlinear terms). Surprisingly, this also applies to most upper triangular input matrices, i.e., when no loops in the material flows exist.

(iii) Another relevant case is sequential supply chains and regular supply networks [see, e.g., Fig. 2(a)]. These are mostly characterized by degenerate zero eigenvalues $J_{ii}=0$ and Jordan normal forms \mathbf{J} , i.e., nonvanishing upper-diagonal elements $J_{i,i+1}$. Sequential supply chains and regular distribution systems belong to this case, which is characterized by the two u -fold degenerate eigenvalues

$$\lambda_{\pm} = -(1+B)/2 \pm \sqrt{(1+B)^2/4 - A} \quad (38)$$

independently of the suppliers i . For small enough values $A < (1+B)^2/4$, these systems show overdamped behavior, otherwise damped oscillations.

Note that already very small perturbations in the network structure can qualitatively change the dynamics of supply networks. In case (iii), for example, they cause a cluster of complex eigenvalues around the two negative eigenvalues λ_{\pm} (if A is small, see Fig. 5). How can we explain this interesting observation? We may use *Geršgorin's theorem* on the location of eigenvalues [17]. Applying it to the perturbed matrix

$$\mathbf{B}_{\eta} = \mathbf{E} - \mathbf{S} + \eta\mathbf{P}, \quad (39)$$

for small enough values of η , the corresponding eigenvalues should be located within disks of radius

$$R_i(\eta) = \eta \sum_j |P_{ij}^{(\eta)}| \quad (40)$$

around the eigenvalues J_{ii} of the original matrix $\mathbf{E} - \mathbf{S}$. This radius grows monotonously, but not necessarily linearly in the parameter η with $0 < \eta \leq 1$, which allows to control the size of the perturbation. Moreover, $\mathbf{P}^{(\eta)} = \mathbf{R}_{\eta}^{-1} \mathbf{P} \mathbf{R}_{\eta}$, where \mathbf{R}_{η} is the orthogonal matrix which transforms \mathbf{B}_{η} to a diagonal matrix $\mathbf{D}^{(\eta)}$, i.e., $\mathbf{R}_{\eta}^{-1} \mathbf{B}_{\eta} \mathbf{R}_{\eta} = \mathbf{D}^{(\eta)}$. (This assumes a perturbed matrix \mathbf{B}_{η} with no degenerate eigenvalues.) Similar disks as for the eigenvalues of \mathbf{B}_{η} can be determined for the associated eigenvalues $\lambda_{i,\pm}^{(\eta)}$ of the perturbed $(2u \times 2u)$ matrix \mathbf{M}_{η} belonging to the perturbed $u \times u$ matrix \mathbf{B}_{η} [see Eq. (13) and Fig. 5].

In summary, this contribution has shown how supply systems can be modeled by equations for material flows in networks. We could explain the empirically observed bullwhip effect as a convective instability phenomenon based on a resonance effect and generalize it from sequential supply chains to arbitrary supply networks, as these can be transformed to supply chains for “quasisuppliers.” However, the eigenvalues may become complex. The dynamics of supply networks depends very sensitively on their topology and two parameters A and B , which are related to derivatives of the “control function” $W_i(N_i, dN_i/dt)$ of the delivery rates. In contrast to most systems of mechanical or electrical oscillators, supply networks have directed links and are asymmetric. Therefore, most supply networks are expected to show a damped or growing oscillatory behavior, even without loops. Although negative eigenvalues are found for regular distribution networks and supply chains, already small perturbations of the network structure will cause a spectrum of complex eigenvalues, i.e., a qualitative change in the dynamics. Our current research focuses on the application to empirical input-output data and business cycles [13], on nonlinear properties of supply networks, and applications to disaster management with a lack of resources [7].

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- [14] The same equations result from a more detailed supply chain model in cases where there is no lack of resources [3]. Moreover, for $\mathbf{D}=(\delta_{ij})$ they correspond to the linearized equations of a macroeconomic model of commodity flows, if the additional price adjustment assumed in this model is slow [13]. Furthermore, Eq. (12) can reflect management strategies adapting the delivery rates to anticipated inventories $N_i(t+\Delta t) \approx N_i(t) + \Delta t dN_i/dt$, as suggested in Ref. [4]. BT is then proportional to the anticipation time horizon Δt .
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